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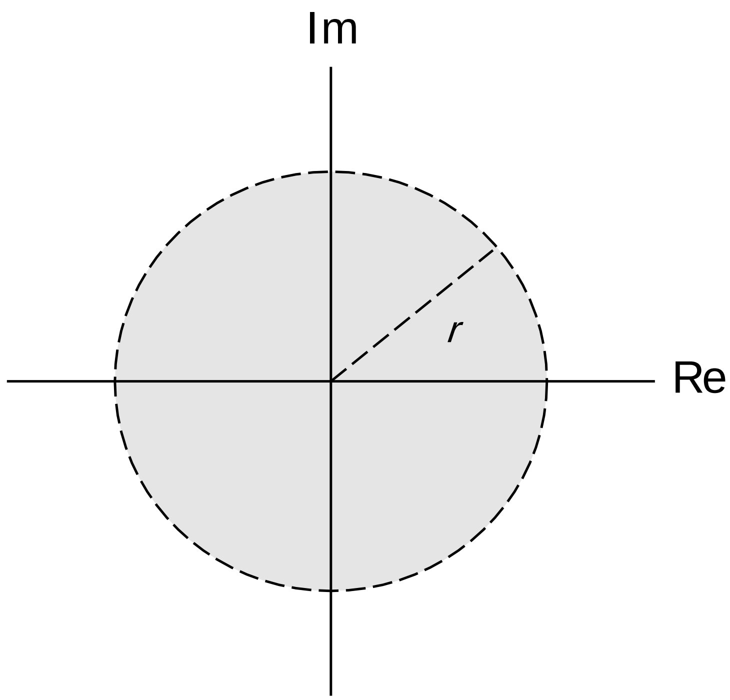
Section 11.2

Region of Convergence (ROC(

- A disc with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| < r,$$

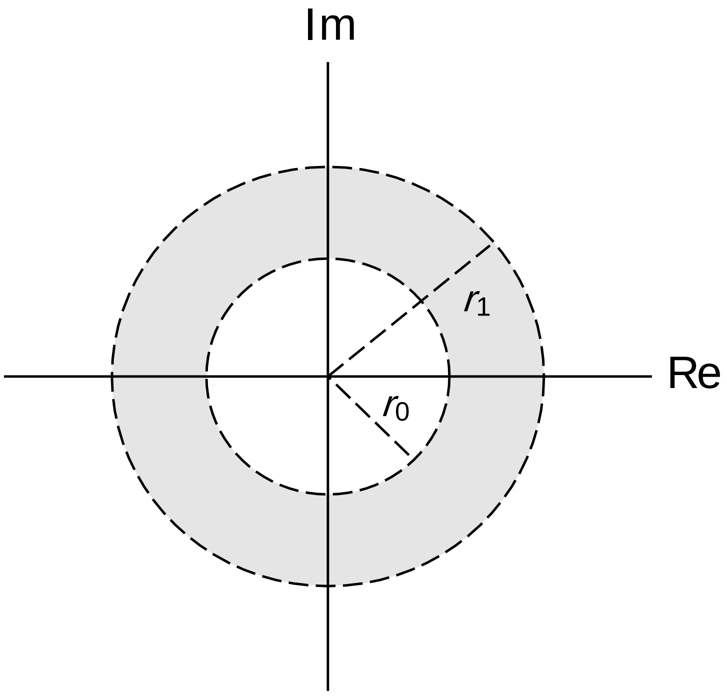
where r is a real constant and $r > 0$.



- An annulus with center 0, inner radius r_0, and outer radius r_1 is the set of all complex numbers z satisfying

$$r_0 < |z| < r_1,$$

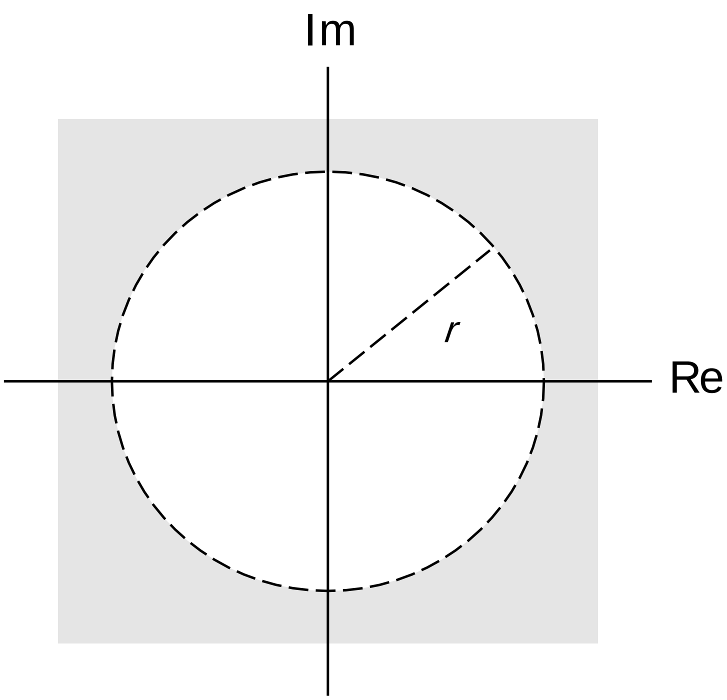
where r_0 and r_1 are real constants and $0 < r_0 < r_1$.



- The exterior of a circle with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| > r,$$

where r is a real constant and $r > 0$.



1. The ROC consists of *concentric circles centered at the origin* in the complex plane.
2. If the sequence x has a *rational* z transform, then the ROC *does not contain any poles*, and the ROC is *bounded by poles or extends to infinity*.
3. If the sequence x is *finite duration*, then the ROC is the *entire complex plane*, except possibly the origin (and/or infinity.(
4. If the sequence x is *right sided* and the circle $|z| = r_0$ is in the ROC, then all (finite) values of z for which $|z| > r_0$ will also be in the ROC (i.e., the ROC contains all points *outside the circle*.(
5. If the sequence x is *left sided* and the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC (i.e., the ROC contains all points *inside the circle*, except possibly the origin.(
6. If the sequence x is *two sided* and the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring that includes this circle (i.e., the ROC is an *annulus* centered at the origin containing the circle.(

7. If the z transform X of x is *rational* and x is *right sided*, then the ROC is the region outside the outermost pole (i.e., outside the circle of radius equal to the largest magnitude of the poles of X). (If x is causal, then the ROC also includes infinity.)
8. If the z transform X of x is *rational* and x is *left sided*, then the ROC is the region inside the innermost nonzero pole (i.e., inside the circle of radius equal to the smallest magnitude of the nonzero poles of X and extending inward to and possibly including the origin). If x is anticausal, then the ROC also includes the origin.

- Some of the preceding properties are redundant (e.g., properties 1, 2, and 4 imply property 7).
- The ROC must always be of the form of one of the following:
  1. a disc centered at the origin, possibly excluding the origin
  2. an annulus centered at the origin
  3. the exterior of a circle centered at the origin (possibly excluding infinity)
  4. the entire complex plane, possibly excluding the origin (and/or infinity)

Section 11.3

Properties of the Z Transform

Property	Time Domain	Z Domain	ROC
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least $R_1 \cap R_2$
Translation	$x(n - n_0)$	$z^{-n_0} X(z)$	R except possible addition/deletion of 0
Z-Domain Scaling	$a^n x(n)$	$X(a^{-1} z)$	$\lvert a \rvert R$
	$e^{j\Omega_0 n} x(n)$	$X(e^{-j\Omega_0} z)$	R
Time Reversal	$x(-n)$	$X(1/z)$	R^{-1}
Upsampling	$(\uparrow M) x(n)$	$X(z^M)$	$R^{1/M}$
Conjugation	$x^*(n)$	$X^*(z^*)$	R
Convolution	$x_1 * x_2(n)$	$X_1(z) X_2(z)$	At least $R_1 \cap R_2$
Z-Domain Diff.	$nx(n)$	$-z \frac{d}{dz} X(z)$	R
Differencing	$x(n) - x(n-1)$	$(1 - z^{-1}) X(z)$	At least $R \cap \lvert z \rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{z}{z-1} X(z)$	At least $R \cap \lvert z \rvert > 1$

Property	
Initial Value Theorem	$x(0) = \lim_{z \to \infty} X(z)$
Final Value Theorem	$\lim_{n \to \infty} x(n) = \lim_{z \to 1} [(z-1) X(z)]$

Pair	$x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
3	$-u(-n-1)$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$-nu(-n-1)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert < 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$-a^n u(-n-1)$	$\frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$-na^n u(-n-1)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
10	$(\cos\Omega_0 n)u(n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\lvert z\rvert > 1$
11	$(\sin\Omega_0 n)u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\lvert z\rvert > 1$
12	$(a^n\cos\Omega_0 n)u(n)$	$\frac{z(z-a\cos\Omega_0)}{z^2-2az\cos\Omega_0+a^2}$	$\lvert z\rvert > \lvert a\rvert$
13	$(a^n\sin\Omega_0 n)u(n)$	$\frac{az\sin\Omega_0}{z^2-2az\cos\Omega_0+a^2}$	$\lvert z\rvert > \lvert a\rvert$

- If $x_1(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)$ with ROC R_1 and $x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_2(z)$ with ROC R_2, then

  $a_1x_1(n) + a_2x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z)$ with ROC R containing $R_1 \cap R_2$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the linearity property of the z transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs.(

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$x(n-n_0) \xleftrightarrow{\text{ZT}} z^{-n_0}X(z), \quad \text{with ROC } R'$$

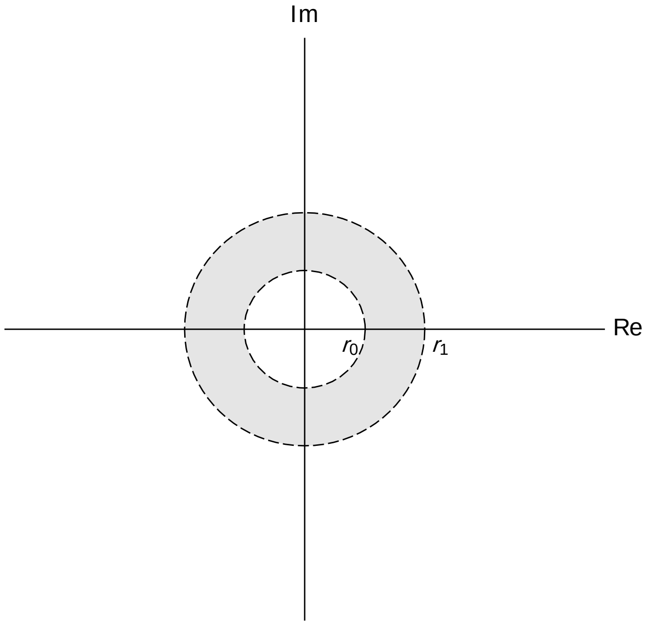
- •where n_0 is an integer constant and R' is the same as R except for the possible addition or deletion of zero or infinity.

  •This is known as the translation (or time-shifting) property of the z transform.

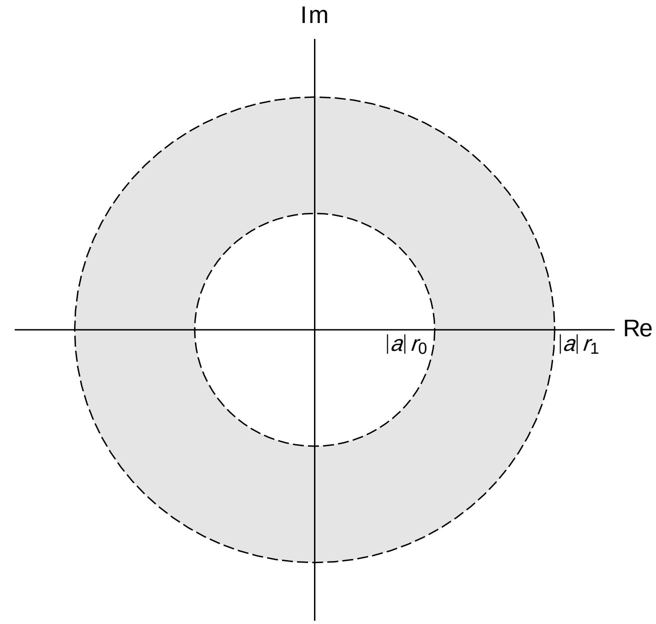
- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$a^n x(n) \xleftrightarrow{\text{ZT}} X(z/a) \quad \text{with ROC } |a|\,R,$$

where a is a nonzero constant.
- This is known as the z-domain scaling property of the z transform.
- As illustrated below, the ROC R is *scaled* by $|a|$.



R

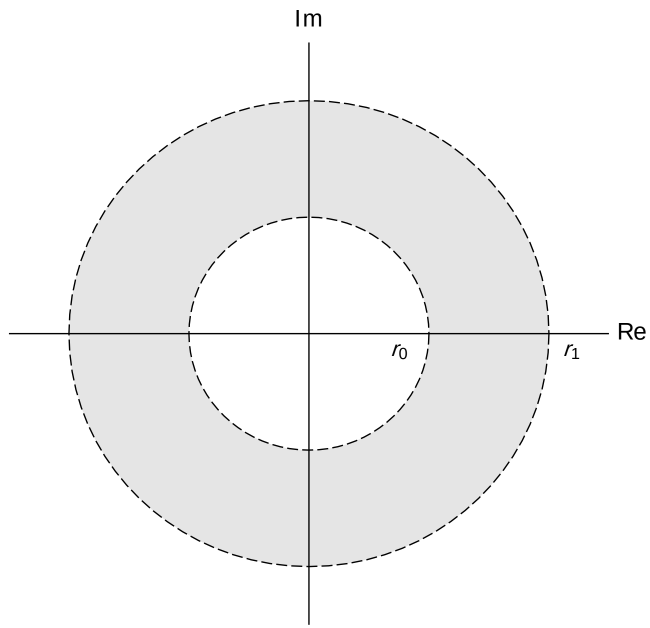


$|a|\,R$

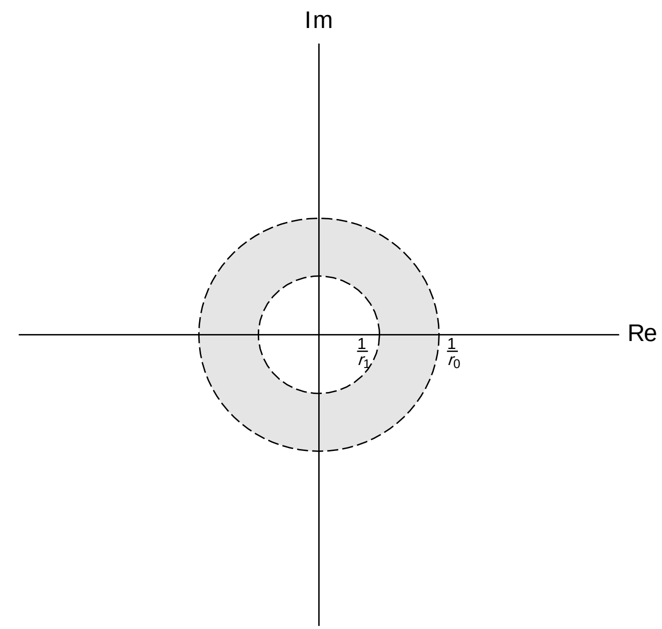
- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(-n) \overset{\text{ZT}}{\longleftrightarrow} X(1/z) \quad \text{with ROC } 1/R.$$

- This is known as the time-reversal property of the z transform.
- As illustrated below, the ROC R is *reciprocated*.



R



$/\,1R$

- Define $(\uparrow M)x(n)$ as

$$(\uparrow M)x(n) = \begin{cases} x(n/M) & \text{if } n/M \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$(\uparrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z^M) \quad \text{with ROC } R^{1/M}.$$

- This is known as the upsampling (or time-expansion) property of the z transform.